Tutorial 6 : Selected problems of Assignment 7

Leon Li

31/10/2018

Recall the Contraction Mapping Principle:

Thm Let (X,d) be a complete metric space, $T: X \to X$ be a contraction

Then there exists a unique fixed point y of T such that

for any $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}} := (T^n x_0)$ converges to y.

Q1) (HW 7, Q10)

Fix $\alpha \in [0,1)$, for each $x_0 \in [0,1]$, consider the iteration

$$x_n := \alpha x_{n-1}(1-x_{n-1}), \quad n \in \mathbb{N}$$

(a) Show that $(x_n) \subseteq [0,1]$

(b) Show that $\lim_n x_n = 0$

Sol: (a) Define $T : [0,1] \to \mathbb{R}$ by $Tx := \alpha x(1-x)$

Then T is smooth with $T'(x) = \alpha(1-2x)$; $T''(x) = -2\alpha$

$\therefore$ T achieve maximum at $\frac{1}{2}$ with $T(\frac{1}{2}) = \frac{\alpha}{4} < 1$

Clearly, $Tx \geq 0$, $\forall x \in [0,1]$, $\therefore$ $T([0,1]) \subseteq [0,1]$

In particular, $\forall n$, $x_n = T^n x_0 \in [0,1]$ as $x_0 \in [0,1]$

(b) Note that $T(0) = 0$, hence 0 is a fixed point of T.

Also, take $\gamma = \max_{x \in [0,1]} |T'(x)| = \alpha < 1$, then $\forall x, x' \in [0,1]$

$|Tx - Tx'| \leq \gamma \cdot |x - x'|$, $\therefore$ $T : [0,1] \to [0,1]$ is a contraction.

Therefore, $\lim_n x_n = \lim_n T^n x_0 = 0$ by the theorem. $-\boxtimes$

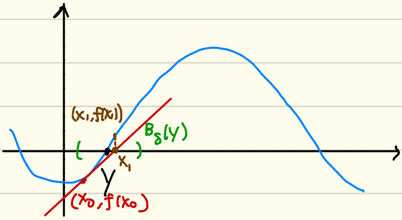
Q2) (HW7, Q9) (Newton's method)

Let $f: \mathbb{R} \to \mathbb{R}$ be C^2, $y \in \mathbb{R}$ such that $f(y) = 0$; $f'(y) \neq 0$.

Show that there exists $\rho > 0$ s.t. $\forall\, x_0 \in \overline{B_\rho(y)} := \{x \in \mathbb{R} \mid |x-y| \leq \rho\}$

the iterated sequence $x_n := x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$, $n \in \mathbb{N}$ converges to y.

Picture:



Sol: As $f'(y) \neq 0$, there exists $\delta > 0$ s.t. $f'(x) \neq 0$, $\forall x \in B_\delta(y)$

Define $T: B_\delta(y) \to \mathbb{R}$ by $Tx := x - \frac{f(x)}{f'(x)}$

then T is differentiable w/ derivative

$$T'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{(f'(x))^2} \text{ is continuous}$$

Hence, T is C^1 with $Ty = y$; $T'(y) = 0$

By continuity of T' at y, there exists $\frac{\delta}{2} > \rho > 0$ s.t.

$$|T'(x)| < 1, \ \forall\, x \in B_{2\rho}(y).$$

Then we claim that $T : \overline{B_\rho(y)} \to \mathbb{R}$ is a contraction:

Choose $\gamma := \max_{x \in \overline{B_\rho(y)}} |T'(x)| < 1$, then $\forall x, x' \in \overline{B_\rho(y)}$,

$$|Tx - Tx'| = |T'(\xi)|\,|x - x'|, \ \exists\, \xi \text{ between } x, x', \text{ by Mean Value Theorem}$$

$$\leq \gamma \cdot |x - x'|$$

In particular, take $x' = y$, then

$$|Tx - y| = |Tx - Ty| < \gamma |x - y| < 1 \cdot \rho = \rho, \ \forall x \in \overline{B_\rho(y)}$$

$\therefore T(\overline{B_\rho(y)}) \subseteq \overline{B_\rho(y)}$, and $T : \overline{B_\rho(y)} \to \overline{B_\rho(y)}$ is a contraction

Since $\overline{B_\rho(y)}$ is complete and y is a fixed point of T,

by uniqueness part of the theorem, $\forall x_0 \in \overline{B_\rho(y)}$,

$$x_n := x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = Tx_{n-1} = T^2 x_{n-2} = \cdots = T^n x_0 \text{ converges to } y.$$

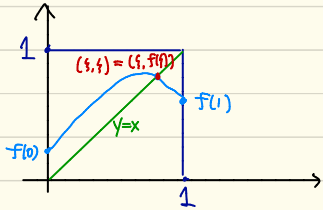
$-\boxtimes$

Q3) (HW7, Q11)

Let $f:[0,1]\to[0,1]$ be a continuous function.

Show that f has a fixed point.

Picture :



Pf: If $f(0)=0$ or $f(1)=1$, then result follows.

Otherwise, assume $f(0)>0$ and $f(1)<1$.

Define $g:[0,1]\to\mathbb{R}$ by $g(x)=f(x)-x$

then $g(0)>0$ and $g(1)<0$.

As g is continuous, by Intermediate Value Theorem,

there exists $\xi\in(0,1)$ s.t. $g(\xi)=0$, i.e. $f(\xi)=\xi$

Therefore, f has a fixed point. $\leftarrow \blacksquare$